Linear systems and Fano varieties: BAB, and related topics

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References:
[B-1] Anti-pluricanonical systems on Fano varieties.
[B-2] Singularities of linear systems and boundedness of Fano varieties.

Singularities of anti-canonical systems

Lets recall the theorem on lc thresholds of anti-canonical systems.

Theorem ([B-2])

For each $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$ there is $t \in \mathbb{R}^{>0}$ such that if

- (X, B) *is a projective ϵ-lc pair of dimension d, and*
- $A := -(K_X + B)$ *is nef and big,*

then

$$\operatorname{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$

Rough idea of proof:

Assume $\operatorname{lct}(X, B, |A|_{\mathbb{R}})$ is too small.

We modify the setting so that X is Fano and $B = 0$, and there is $0 \leq L \sim_{\mathbb{R}} -K_X$ with too large coefficients.

Singularities of anti-canonical systems

Next we show X is birational to a smooth projective and bounded model $\overline{X}$.

Also we find a bounded $\overline{\Gamma}$ so that $(\overline{X}, \overline{\Gamma})$ is ϵ-lc.

We find $\overline{N} \geq 0$ with bounded"degree" but too small lc threshold $\mathrm{lct}(\overline{X}, \overline{\Gamma}, \overline{N})$.

This contradicts our main theorem on boundedness of lc thresholds:

Theorem ([B-2])

For each $d, r \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$ there is $t \in \mathbb{R}^{>0}$ such that if

- *(X, B) is projective ϵ-lc of dimension d,*
- *A is very ample with $A^d \leq r$, and*
- *$A - B$ is ample,*

then

$$\mathrm{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$

Boundedness of Fano varieties: BAB

Lets recall BAB.

Theorem ([B-2])

Assume $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$. Then the set

$$\{X \mid X \ \ \epsilon\text{-lc Fano of dimension } d\}$$

forms a bounded family.

Sketch of proof in dimension 2 following Alexeev-Mori:

For simplicity assume $B = 0$ and that $-K_X$ is ample.

There is $\Delta \geq 0$ such that (X, Δ) is ϵ-lc and $K_X + \Delta \sim_{\mathbb{R}} 0$.

Let $\phi\colon W \to X$ be the minimal resolution and let $K_W + \Delta_W$ be the pullback of $K_X + \Delta$.

Boundedness of Fano varieties: BAB

Since (X, Δ) is klt, the exceptional divisors of ϕ are all smooth rational curves.

Moreover, by basic properties of minimal resolutions, $\Delta_W \geq 0$.

In particular, (W, Δ_W) is an ϵ-lc pair.

Now a simple calculation of intersection numbers shows that $-E^2 \leq l$ for every exceptional curve of ϕ where $l \in \mathbb{N}$ depends only on ϵ.

If the number of exceptional curves of ϕ is bounded, then the Cartier index of $-K_X$ is bounded

which in turn implies $-nK_X$ is very ample for some bounded n.

In particular, this holds if the Picard number of W is bounded from above.

Boundedness of Fano varieties: BAB

If in addition $\mathrm{vol}(-K_X)$ is bounded, then X belongs to a bounded family.

Note that $\mathrm{vol}(-K_X) = \mathrm{vol}(-K_W)$.

Running an MMP on K_W we get a morphism $W \to V$ where V is either $\mathbb{P}^2$ or a rational ruled surface.

and the morphism is a sequence of blowups at smooth points.

Let Δ_V be the pushdown of Δ_W.

Then (V, Δ_V) is ϵ-lc and $K_V + \Delta_V \sim_{\mathbb{R}} 0$.

It is easy to show that there are finitely many possibilities for V.

In particular, from $\mathrm{vol}(-K_W) \leq \mathrm{vol}(-K_V)$, we deduce that $\mathrm{vol}(-K_X) = \mathrm{vol}(-K_W)$ is bounded from above.

Boundedness of Fano varieties: BAB

Thus it is enough to prove that the number of blowups in $W \to V$ is bounded.

This number can be bounded by an elementary analysis of possible intersection numbers in the sequence.

Sketch of proof in higher dimension [B-2]:

First applying a result of [Hacon-Xu], it is enough to show that K_X has a klt strong m-complement for some bounded number $m \in \mathbb{N}$.

Running MMP on $-K_X$ and replacing X with the resulting model we can assume $B = 0$.

By boundedness of complements, there is an lc strong n-complement $K_X + B^+$.

If X is exceptional, the complement is klt, so we are done in this case.

Boundedness of Fano varieties: BAB

To treat the general case the idea is to modify the complement $K_X + B^+$ into a klt one.

We will do this using birational boundedness.

We need to show $\mathrm{vol}(-K_X)$ is bounded from above.

This can be proved using arguments similar to the proof of the effective birationality theorem.

Once we have this bound, we can show that (X, B^+) is log birationally bounded,

that is, there exist a bounded log smooth projective pair $(\overline{X}, \Sigma_{\overline{X}})$ and a birational map $\overline{X} \dashrightarrow X$ such that

$\Sigma_{\overline{X}}$ contains the exceptional divisors of $\overline{X} \dashrightarrow X$ and the support of the birational transform of B^+.

Boundedness of Fano varieties: BAB

Next we pull back $K_X + B^+$ to a high resolution of X and push it down to $\overline{X}$ and denote it by $K_{\overline{X}} + B^+_{\overline{X}}$.

Then $(\overline{X}, B^+_{\overline{X}})$ is sub-lc and $n(K_{\overline{X}} + B^+_{\overline{X}}) \sim 0$.

Now $\operatorname{Supp} B^+_{\overline{X}}$ is contained in $\Sigma_{\overline{X}}$.

So we can use the boundedness of $(\overline{X}, \Sigma_{\overline{X}})$ to perturb the coefficients of $B^+_{\overline{X}}$.

More precisely, perhaps after replacing n, there is $\Delta_{\overline{X}} \sim_{\mathbb{Q}} B^+_{\overline{X}}$

such that $(\overline{X}, \Delta_{\overline{X}})$ is sub-klt and $n(K_{\overline{X}} + \Delta_{\overline{X}}) \sim 0$.

Pulling $K_{\overline{X}} + \Delta_{\overline{X}}$ back to X and denoting it by $K_X + \Delta$ we get a sub-klt (X, Δ) with $n(K_X + \Delta) \sim 0$.

Boundedness of Fano varieties: BAB

Now a serious issue here is that Δ is not necessarily effective.

In fact it is by no means clear that its coefficients are even bounded from below.

This is one of the difficult steps of the proof.

We need to show: if $0 \leq L \sim_{\mathbb{R}} -K_X$, then coefficients of L are bounded from above.

However, this boundedness follows directly from boundedness of lc threshold of anticanonical systems on Fano's (Ambro's conjecture).

In turn this boundedness of thresholds follows from the (general) theorem on boundedness of lc thresholds.

The rest of the argument which modifies Δ to get a klt complement is an easy application of complement theory.

Fano type fibrations

One of the possible outcomes of the MMP is a Mori fibre space which is an extremal contraction $X \to Z$ where K_X is anti-ample over Z.

This is a special kind of Fano fibration.

Fano fibrations and more generally Fano type fibrations appear naturally when studying uniruled varieties, and also in the context of moduli theory.

Theorem (Mori-Prokhorov)

Let X be a 3-fold with terminal singularities, and $f\colon X \to Z$ be a Mori fibre space;

- *if Z is a surface, then Z has canonical sing;*
- *if Z is a curve, then multiplicities of fibres of f are bounded.*

Fano type fibrations

M^{c}Kernan proposed a generalisation of the first part to higher dimension:

Conjecture

Assume $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$. Then there is $\delta \in \mathbb{R}^{>0}$ such that if

- *$f\colon X \to Z$ is a Mori fibre space, and*
- *X is ϵ-lc $\mathbb{Q}$-factorial of dimension d,*

then Z is δ-lc.

This is known in toric case [Alexeev-Borisov].

Independently, Shokurov proposed a more general problem which generalised both parts of Mori and Prokhorov result.

Fano type fibrations

Conjecture (Shokurov)

For each $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$, there is $\delta > 0$ such that if

- (X, B) *is ϵ-lc of dim d, $f\colon X \to Z$ is a contraction,*
- $K_X + B \equiv 0/Z$, $-K_X$ *is big$/Z$,*

then we can write

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

such that $(Z, B_Z + M_Z)$ is δ-lc.

Theorem (B, 2012)

Shokurov conjecture holds if $(F, \operatorname{Supp} B|_F)$ belongs to a bounded family where F is a general fibre.

Note: BAB implies F belongs to a bounded family.

Singularities: minimal log discrepancies

The lc threshold plays an important role in birational geometry.

This is clear from the proofs described in these talks.

It is also related to the termination conjecture [B, 2006].

Another more subtle invariant of singularities is the **minimal log discrepancy** (mld) also defined by Shorkuov.

Let (X, B) be a pair.

The mld of (X, B) is defined as

$$\text{mld}(X, B) := \min\{\text{log discrepancy } \ a(D, X, B)\}$$

where D runs over all prime divisors on birational model of X.

The mld is way harder to treat than the lc threshold.

Singularities: minimal log discrepancies

Shokurov proposed the following:

Conjecture (ACC for mld's)

Assume $d \in \mathbb{N}$ and $\Phi \subset [0, 1]$ is a set of numbers satisfying the descending chain condition (DCC).

Then the set

$$\{\text{mld}(X, B) \mid (X, B) \textit{ is an lc pair and coefficients of } B \textit{ are in } \Phi\}$$

satisfies the ascending chain condition (ACC).

Known in dimension two [Alexeev][Shokurov: unpublished].

Its importance is in relation with the termination conjecture and other topics of interest.

Singularities: minimal log discrepancies

This ACC conjecture together with a semi-continuity conjecture about mld's (due to Ambro) imply the termination conjecture [Shokurov].

The expectation is that the ACC conjecture can be approached using the theory of complements and the methods described in these talks.

Note that a similar ACC statement holds for lc thresholds [Hacon-McKernan-Xu] which can be reproved using BAB.

Stable Fano varieties

Existence of specific metrics, e.g. Kähler-Einstein metrics, on manifolds is a central topic in differential geometry.

Unlike canonically polarised and Calabi-Yau manifolds ([Yau][Aubin-Yau]), Fano manifolds do not always admit such metrics.

A Fano manifold admits a Kähler-Einstein metric iff it is *K-polystable* [Chen-Donaldson-Sun].

On the other hand, Fano varieties do not behave as well as canonically polarised varieties in the context of moduli theory.

For example, the moduli space would not be separated.

A remedy is to consider only *K-semistable* Fano's.

Stable Fano varieties

The first step of constructing a moduli space is to prove a suitable boundedness result.

In the smooth case this is not an issue.

But in the singular case boundedness is a recent result.

Using methods described in these talks, Jiang proved such a result.

He showed the set of K-semistable Fano varieties X of fixed dimension and $\mathrm{vol}(-K_X)$ bounded from below forms a bounded family.

Other topics

There are connections with other topics of interest not discussed above.

Here we only mention two works very briefly.

Lehmann-Tanimoto-Tschinkel and Lehmann-Tanimoto relate boundedness of Fano's and related invariants to the geometry underlying Manin's conjecture on distribution of rational points on Fano varieties.

One the other hand, Di Cerbo-Svaldi studied boundedness of Calabi-Yau pairs where boundedness of Fano varieties appear naturally.